Quadratic Variance Swap Models

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ÉCOLE POLYTECHNIQUE
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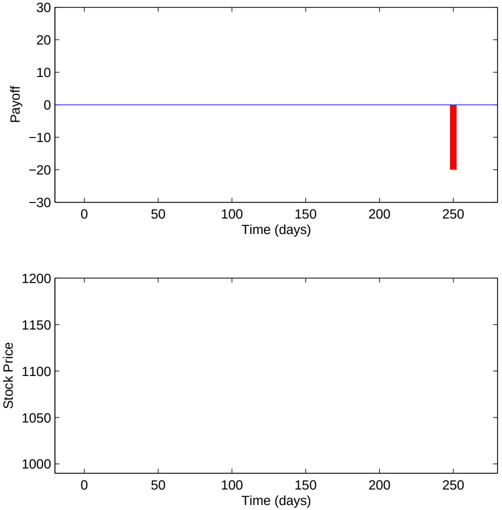
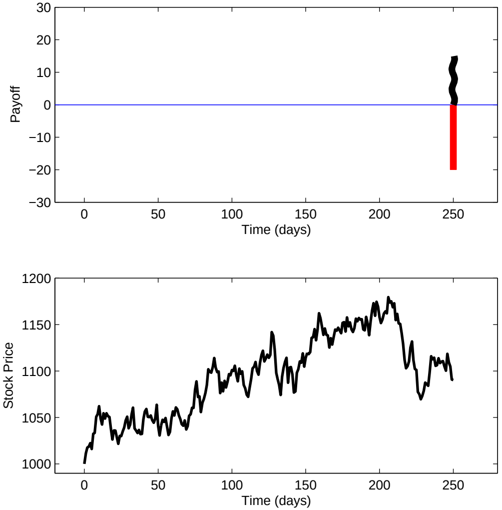
Variance Swap (time 0)



Variance Swap (time T)



Outline

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Motivation

Variance Swap

Quadratic Model

Optimal Investment

Volatility Derivatives

Over last two decades, large demand of volatility derivatives:
variance swaps, volatility swaps, corridor integrated variance,. . .

Variance swaps: traded over-the-counter

- on various underlying assets (equity indices, exchange/interest rates, commodities, etc.)
- at many different maturities ($\Rightarrow$ term structure)

Why Variance Swaps?

Variance swaps have two distinctive features:

1. **Easier** to hedge than other volatility derivatives:
   static position in options and dynamic trading of futures;
   Dupire (1993), Neuberger (1994), Carr and Madan (1998), a.o.

2. **Direct** exposure to variance risk over a fixed time horizon.
   CBOE futures and options on VIX *not* equally direct exposure
   - VIX index (30-day S&P 500 volatility index)
   - introduced in 1993 (back-calculated to 1990, revised in 2003)
   - 3/2004 futures on VIX
   - 2/2006 European options on VIX
   - 12/2012 "S&P 500 Variance Futures"

Wall Street Journal, 22 October 2008

22 WEDNESDAY, OCTOBER 22, 2008 THE WALL STREET JOURNAL.

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Volatility kills market for

Traders in the swaps can't hedge against unforeseen extremes

BY RENÉE S

The payoff, which is the difference between these two amounts, is paid when the contract expires

The swaps in the

this centu

volatility swaps

Changing times

The rise in volatility has been so abrupt that it has frozen the market for volatility-related derivatives. Year-to-date performance of CBOE Volatility Index (VIX)

Tuesday midafternoon: 54.42

Source: Thomson Reuters Datastream

They use the options markets to hedge the swaps and a deterioration in liquidity in single stock options hasn't helped. In addition, few were prepared with their hedges to face the extreme levels of volatility of recent weeks.

Furthermore, these products have caps that set the maximum pay out. Many of the swaps have hit this maximum limit, but they are staying alive until maturity. By that time, the value of the pay out might have fallen under the cap if volatility has declined.

"The most important thing ... swap versus a nor- ... options

behave like stocks," said Simon Yates, global head of equity derivatives at Credit Suisse in London.

"Variance swaps behave like options all the time because there's no strike, so even if you move down 10% they're alive and well," he said. "So upfront you need to make an assumption about what the volatility should be at all different spot levels, so 10%, 20%, 50%, 70% and so on. It is not really complexity, but how you get those inputs about very deep out-of-the money volatilities into your model."

Dealers pointed to Dutch-Belgian bank Fortis NV , which this month was taken over by its state governments and its shares suspended from trading, only to reopen down more than 60% last week when trading was resumed.

To get around the problem and hedge the risk associated with the variance swaps that they can't get off their books, some dealers have been packaging the risk into new products to offset it.

"We've seen a couple of banks scrambling to cover single stock variance through dispersion trades," Mr. Yates said. "So as a package they've gone to hedge fund clients and bought single stock variance versus selling index variance as one trade. Then they take the index they've sold and buy it back in a separate trade because the index is still liquid."

While some shell-shocked dealers believe variance swaps have had their day, others are convinced they will re-emerge as a more robust product.

"These are very useful products for hedge funds, in the sense that if they want to take a pur... view on realized volatility, var... ance is the cheapest product ar... they're easy to keep track of ... they only have one payment at ... ," Mr. Yates said. "For th... always going t...

Contribution

1. Novel and flexible model for term structure of variance swaps

2. Dynamic optimal investment in variance swaps, S&P 500, index option, and risk-free bond

Outline

Setup

- S_t index price (S&P 500), under $\mathbb{Q}$:

$$\frac{dS_t}{S_{t-}} = r_t\,dt + \sigma_t\,dB_t + \int_{\mathbb{R}} \xi\,(\chi(dt,d\xi) - \nu_t(d\xi)dt)$$

- Quadratic variation over horizon $[t, t+\tau]$:

$$\mathrm{QV}(t,t{+}\tau) = \frac{1}{\tau}\left(\int_t^{t+\tau} \sigma_s^2\,ds + \int_t^{t+\tau}\int_{\mathbb{R}} (\log(1+\xi))^2 \chi(ds,d\xi)\right)$$

- Variance Swap payoff:

$$\mathrm{QV}(t,t+\tau) - \mathrm{VS}(t,t+\tau)$$

- Variance Swap rate (depends on $\tau \Rightarrow$ term structure):

$$\mathrm{VS}(t,t+\tau) = \mathbb{E}_t^{\mathbb{Q}}[\mathrm{QV}(t,t+\tau)]$$

Variance Swap Data set

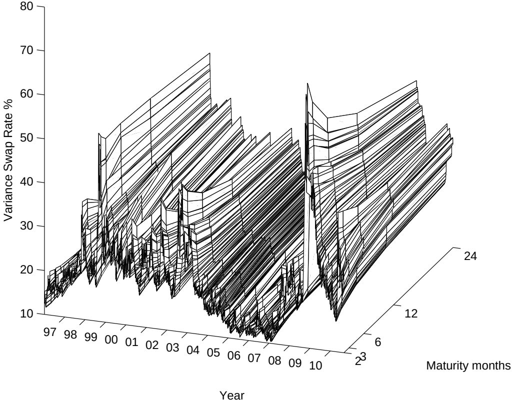


Figure: Variance swap rates, $\sqrt{\mathrm{VS}(t, t+\tau)} \times 100$, on the S&P 500 index from 4-Jan-1996 to 2-Sep-2010, daily quotes. Source: Bloomberg.

Summary Statistics

τ	Mean	Std	Skew	Kurt
Variance Swap rates				
2	22.14	8.18	1.53	7.08
3	22.32	7.81	1.32	6.05
6	22.87	7.40	1.10	4.97
12	23.44	6.88	0.80	3.77
24	23.93	6.48	0.57	2.92
Realized Variances				
2	18.90	12.40	4.31	28.40
3	19.06	12.04	3.80	21.81
6	19.46	11.33	2.93	13.17
12	20.13	10.47	1.97	6.86
24	20.60	8.81	1.09	3.48

Table: Daily data from 4-Jan-1996 to 2-Sep-2010. Volatility percentage units.

Outline

Quadratic Variance Swap Model

- X is m-dimensional diffusion state process:

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t$$

- X is quadratic if

$$\mu(x) = b + \beta x$$

$$\Sigma(x)\Sigma(x)^\top = a + \sum_{k=1}^{m} \alpha^k x_k + \sum_{k,l=1}^{m} A^{kl} x_k x_l$$

- Define spot variance

$$v_t = \mathrm{VS}(t,t) = \sigma_t^2 + \int_{\mathbb{R}} (\log(1+\xi))^2 \nu_t(d\xi)$$

- A **quadratic variance swap model** is obtained when

$$v_t = \phi + \psi^\top X_t + X_t^\top \pi X_t$$

Term Structure of Variance Swaps

Quadratic variance swap model admits a quadratic term structure:

$$\mathrm{VS}(t, T) = \mathbb{E}_t^{\mathbb{Q}}[\mathrm{QV}(t, T)] = \frac{1}{T - t} G(T - t, X_t)$$

with $G(\tau, x) = \Phi(\tau) + \Psi(\tau)^\top x + x^\top \Pi(\tau) x$

and Φ, Ψ and Π satisfy a linear system of ODEs.

Model Selection

Do we need the quadratic feature?

Data: Daily variance swap rates, and quadratic variation from intraday futures returns

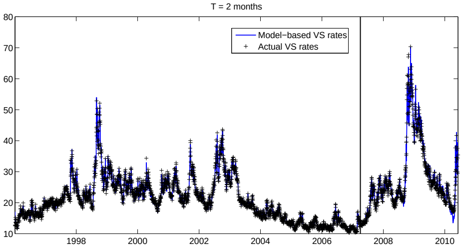
- In-sample (pre-crisis): Jan 4, 1996 to Apr 2, 2007
- Out-of-sample: Apr 3, 2007 to Jun 7, 2010

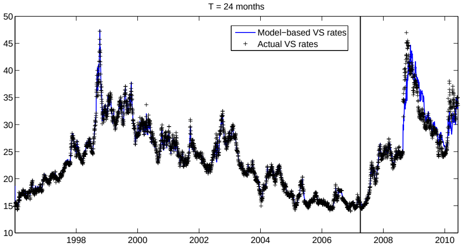
Method: Maximum Likelihood with Unscented Kalman filter

Estimation results:

- Good fit of the *bivariate* quadratic model (likelihood tests, AIC and BIC criteria, pricing errors, forecasting power)
- Somewhat better than affine model with jumps

Fitting Variance Swap Rates





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Optimal Portfolio Problem

Maximize expected utility from terminal wealth V_T of a power utility investor with constant relative risk aversion (CRRA) η

$$\max_{\mathbf{n}_t, w_t, \phi_t, 0 \leq t \leq T} \mathbb{E}^{\mathbb{P}} \left[\frac{V_T^{1-\eta}}{1-\eta} \right]$$

By *dynamically* and *optimally* investing:

- $\mathbf{n}_t = (n_{1t}, \ldots, n_{nt})^\top$ relative notional exposures to each on-the-run τ_i-**variance swap**, $i = 1, \ldots, n$
- w_t fraction of wealth invested in **stock index**
- ϕ_t fraction of wealth invested in **index option**
- and **risk-free bond**

Investing in a Variance Swap

- Variance swap issued at t^* with maturity $T^* = t^* + \tau$
- **Spot value** Γ_t at date $t \in [t^*, T^*]$ of a one dollar notional long position in this variance swap:

$$\begin{aligned}\Gamma_t &= \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-r(T^*-t)}\frac{1}{\tau}\left(\int_{t^*}^{T^*} v_s\,ds - \tau VS(t^*, T^*)\right)\right] \\ &= \frac{\mathrm{e}^{-r(T^*-t)}}{\tau}\left(\int_{t^*}^{t} v_s\,ds + (T^*-t)VS(t, T^*) - \tau VS(t^*, T^*)\right)\end{aligned}$$

- Extends to τ-variance swaps issued at a sequence of inception dates $0 = t_0^* < t_1^* < \cdots$, with $t_{k+1}^* - t_k^* \leq \tau$. At any date $t \in [t_k^*, t_{k+1}^*)$ the investor takes a position in the respective **on-the-run** τ-variance swap with maturity $T^*(t) = t_k^* + \tau$.

Investing in an Index Option

- Assume: index price jumps by a deterministic size $\xi > -1$
- One index option needed to *complete* the market, with price $O_t = O(S_t, X_t)$. The $\mathbb{Q}$-dynamics of O_t

$$
\begin{aligned}
dO_t = & r\, O_t\, dt + \left(\partial_s O_t\, S_t \sigma(X_t) \mathbf{R}(X_t)^\top + \nabla_x O_t^\top \Sigma(X_t)\right) dW_t \\
& + \Delta O_t \left(dN_t - \nu^{\mathbb{Q}}(X_t)\, dt\right)
\end{aligned}
$$

- Index put option in our empirical analysis

Wealth Dynamics

- Resulting wealth process has $\mathbb{Q}$-dynamics

$$\begin{aligned}\frac{dV_t}{V_{t-}} &= \mathbf{n}_t^\top d\mathbf{\Gamma}_t + w_t \frac{dS_t}{S_{t-}} + \phi_t \frac{dO_t}{O_{t-}} + (1 - \mathbf{n}_t^\top \mathbf{\Gamma}_t - w_t - \phi_t)\, r\, dt \\ &= r\, dt + \theta_t^{W\top} dW_t + \theta_t^N \xi \left(dN_t - \nu^{\mathbb{Q}}(X_t)\, dt\right)\end{aligned}$$

- θ_t^W and θ_t^N are defined by $\begin{pmatrix} \theta_t^W \\ \theta_t^N \end{pmatrix} = \mathcal{G}_t \begin{pmatrix} \mathbf{n}_t \\ w_t \\ \phi_t \end{pmatrix}$ with

$$\mathcal{G}_t = \begin{pmatrix} \Sigma(X_t)^\top & \sigma(X_t)\mathbf{R}(X_t) & \mathbf{0}_{d\times 1} \\ \mathbf{0}_{1\times m} & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{D}_t & \mathbf{0}_{m\times 1} & \frac{\nabla_x O_t}{O_{t-}} \\ \mathbf{0}_{1\times n} & 1 & \frac{\partial_s O_t S_t}{O_{t-}} \\ \mathbf{0}_{1\times n} & 1 & \frac{\Delta O_t}{\xi O_{t-}} \end{pmatrix}$$

and $\mathcal{D}_t$ is the $m \times n$ matrix whose ith column is given by $\left(\mathrm{e}^{-r(T_i^*(t)-t)}/\tau_i\right) \nabla_x G(T_i^*(t) - t, X_t)$

Optimal Portfolio Problem

- Maximize expected utility from terminal wealth V_T of a power utility investor with constant relative risk aversion (CRRA) η

$$\max_{\mathbf{n}_t, w_t, \phi_t, 0 \le t \le T} \mathbb{E}^{\mathbb{P}} \left[\frac{V_T^{1-\eta}}{1-\eta} \right]$$

- Pricing kernel:

$$\frac{d\pi_t}{\pi_{t-}} = -r\,dt - \Lambda(X_t)^\top dW_t^{\mathbb{P}} + \left(\frac{\nu^{\mathbb{Q}}(X_t)}{\nu^{\mathbb{P}}(X_t)} - 1 \right) (dN_t - \nu^{\mathbb{P}}(X_t)dt)$$

- Assumption: The market is **complete** with respect to *stock index*, *index option*, and n on-the-run τ_i-*variance swaps*. Thus, $n = m = d - 1$, and the $(d+1) \times (d+1)$ matrix $\mathcal{G}_t$ is **invertible** $dt \otimes d\mathbb{Q}$-a.s.

Optimal Portfolio Problem: Solution via HJB

$$0 = \max_{\theta^W,\,\theta^N} \Bigg\{ \frac{\partial J}{\partial t} + \frac{\partial J}{\partial v} v \left(r + \theta^{W\top}\Lambda(x) - \theta^N \xi \nu^{\mathbb{Q}}(x) \right) + \frac{1}{2}\frac{\partial^2 J}{\partial v^2} v^2 \theta^{W\top}\theta^W$$

$$+ \nabla_x J^\top (\mu(x) + \Sigma(x)\Lambda(x)) + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 J}{\partial x_i \partial x_j} \left(\Sigma(x)\Sigma(x)^\top \right)_{ij}$$

$$+ \theta^{W\top} v \Sigma(x)^\top \nabla_x \left(\frac{\partial J}{\partial v} \right) + \left(J(t, v(1+\theta^N \xi), x) - J(t, v, x) \right) \nu^{\mathbb{P}}(x) \Bigg\}$$

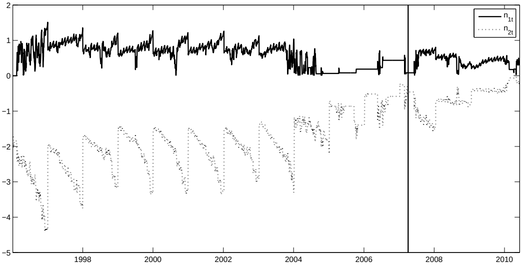
Optimal Allocation: There exists an optimal strategy $\mathbf{n}_t^*$, w_t^*, ϕ_t^* recovered from:

$$\theta_t^{W*} = \frac{1}{\eta}\Lambda(X_t) + \Sigma(X_t)^\top \nabla_x h(T-t, X_t)$$

$$\theta_t^{N*} = \frac{1}{\xi}\left(\left(\frac{\nu^{\mathbb{P}}(X_t)}{\nu^{\mathbb{Q}}(X_t)} \right)^{1/\eta} - 1 \right)$$

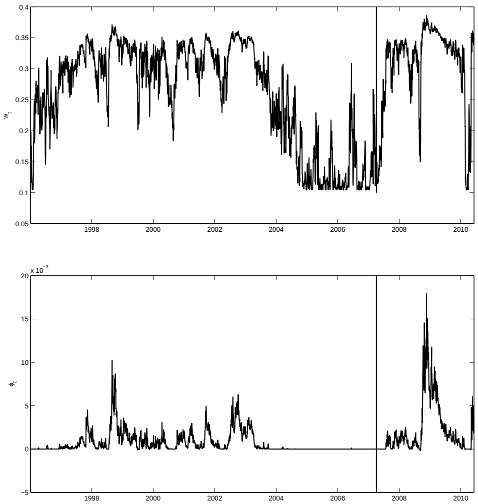
where h is such that e^h satisfies a known PDE

Optimal Investment in VS: Short-Long Strategy



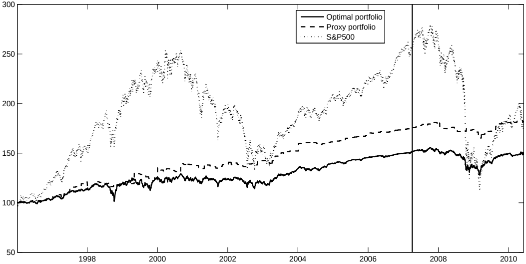
- **Short position** in 2-year VS (earn variance risk premium), **long position** in 3-month VS (hedge volatility risk)
- Periodic patterns in $\mathbf{n}_t$
- Based on bivariate quadratic model

Optimal Investment in Stock Index and Put Option



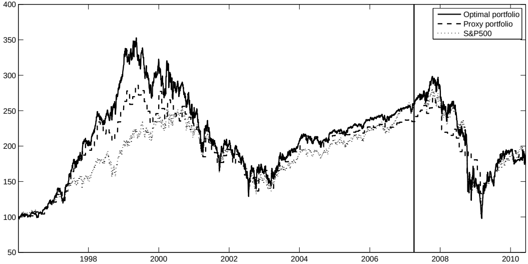
- Positive optimal weight w_t in stock index
- Positive, *tiny* optimal weight ϕ_t in put option

Wealth Trajectory with Optimal Investment



- *Smooth* wealth growth with little volatility
- Suited for risk-averse investors (CRRA $\eta = 5$)
- "Proxy" portfolio (infrequently rebalanced) performs similarly to optimal portfolio (daily rebalanced)

Wealth Trajectory with Optimal Investment: Log-investor



- Larger fluctuations than S&P 500, to seek risk premia
- Suited for less risk-averse investors (CRRA $\eta = 1$)
- "Proxy" portfolio (infrequently rebalanced) performs similarly to optimal portfolio (daily rebalanced)

Conclusion

- Introduce a quadratic term structure model for variance swaps
- Analytically tractable (closed form curves, and explicit conditional moments)
- Optimal investment in variance swaps, stock index, index option, and risk-free bond
- Optimal trading strategy in quasi closed-form:
  - Main feature short-long strategy in variance swaps, i.e., "trading the spread of variance swaps"
  - Stable wealth growth, or more exposure to risk factors (to earn risk premiums), depending on the risk profile of investor